Calculus - Lecture 8

Functions of several real variables. Limits and continuity.

EVA KASLIK

THE VECTOR SPACE $\mathbb{R}^n$

$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) | x_i \in \mathbb{R}^1, i = 1, 2, ..., n\}$.
The elements of $\mathbb{R}^n$ are called vectors.
$\mathbb{R}^n$ is a n-dimensional vector space with respect to the sum and the multiplication by a scalar defined by:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

$$k(x_1, x_2, \ldots, x_n) = (kx_1, kx_2, \ldots, kx_n)$$

For $x \in \mathbb{R}^n$ the norm (or length) of x is defined by

$$\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

The distance between x and $a = (a_1, a_2, \ldots, a_n)$ is $\|x - a\|$.
A neighborhood of $a \in \mathbb{R}^n$ is a set $V \subset \mathbb{R}^n$ which contains a hypersphere $S_r(a)$ centered in a,

$$S_r(a) = \{x \in \mathbb{R}^n \mid \|x - a\| < r\} \qquad r > 0$$

SEQUENCES IN $\mathbb{R}^n$

A sequence (x_k) of vectors of $\mathbb{R}^n$ is a function whose domain of definition is $\mathbb{N}$ and whose values belong to $\mathbb{R}^n$.

A vector $x \in \mathbb{R}^n$ is called the limit of the sequence (x_k) if

$$\forall\, \varepsilon > 0, \exists\, N = N(\varepsilon) > 0 \text{ s.t. } \|x_k - x\| < \varepsilon,\ \forall k \geq N.$$

In this case we write $\lim\limits_{k\to\infty} x_k = x$.

Example. $x_k = (x_k^1, x_k^2) = \left(\frac{1}{k}, \frac{1}{k^2}\right)$ is a sequence in $\mathbb{R}^2$.

Its limit is computed as follows:

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} \left(\frac{1}{k}, \frac{1}{k^2}\right) = (0, 0).$$

Properties

- If the limit of the sequence (x_k) exists, then it is unique.
- If a sequence (x_k) converges to x, then the sequence is bounded: $\exists\, M > 0\, s.t.\, \|x_k\| < M,\ \forall k \in \mathbb{N}$.
- If a sequence (x_k) converges to x, then any subsequence (x_{k_l}) of the sequence (x_k) converges to x.
- Component-wise convergence
  A sequence (x_k), $x_k = (x_{1k}, x_{2k}, ..., x_{nk}) \in \mathbb{R}^n$ converges to $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ if and only if the sequence (x_{ik}) converges to x_i for any $i = 1, 2, ..., n$.
- Bolzano-Weierstrass Theorem
  Any bounded sequence (x_k) of points of $\mathbb{R}^n$ contains a convergent subsequence.
- Cauchy's criterion for convergence
  A sequence $(x_k) \subset \mathbb{R}^n$ converges if and only if for any $\varepsilon > 0$ there exists N_ε such that for $p, q > N_\varepsilon$ we have $\|x_p - x_q\| < \varepsilon$.

FUNCTIONS OF SEVERAL VARIABLES

A real valued function of n variables associates to every vector $x \in D \subset \mathbb{R}^n$ a unique real number.

Formally, $f : D \subset \mathbb{R}^n \to \mathbb{R}^1$ is given by

$$x = (x_1, x_2, \ldots, x_n) \in D \mapsto f(x) = f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$$

Example. $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1 x_2 + \cos(x_1 + x_2)$$

is a real valued function of 2 real variables.

FUNCTIONS OF SEVERAL VARIABLES

A vector valued function of n variables associates to every vector $x \in D \subset \mathbb{R}^n$ a unique vector $f(x)$ from $\mathbb{R}^m$.

Formally, $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$x = (x_1, x_2, ..., x_n) \in D \mapsto f(x) = (f_1(x_1, x_2, \ldots, x_n), \ldots, f_m(x_1, x_2, \ldots, x_n)) \in \mathbb{R}^m$$

The functions $f_i : D \subset \mathbb{R}^n \to \mathbb{R}^1$, $i = \overline{1, m}$, are called scalar components of the vector function f.

Example. The function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$f(x_1, x_2) = (x_1 x_2, x_1 + x_2, \sin(x_1))$$

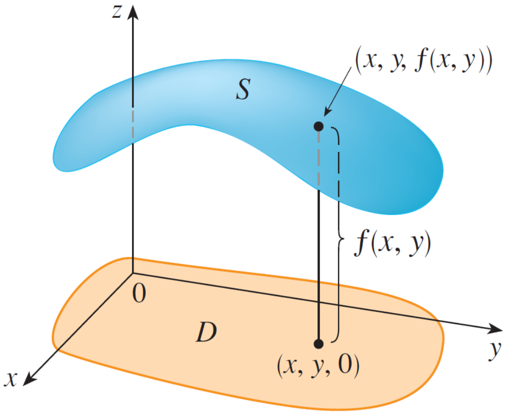
is a vector valued function of 2 real variables.

Its scalar components are:

$$f_1(x_1, x_2) = x_1 x_2$$
$$f_2(x_1, x_2) = x_1 + x_2$$
$$f_3(x_1, x_2) = \sin(x_1)$$

Graphs of real valued functions of two real variables

The graph of function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$



LIMITS

Example. Let's compare the behavior of the functions

$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2} \quad \text{and} \quad g(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

as x and y both approach 0, meaning that $(x,y) \to (0,0)$.

Table 1 Values of $f(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
−0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
−0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Table 2 Values of $g(x, y)$

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

$$\lim_{(x,y)\to(0,0)} f(x,y) = 1 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} g(x,y) \text{ does not exist.}$$

LIMITS

Let us consider $f : D \subset \mathbb{R}^n \to \mathbb{R}^1$ a real valued function of n variables and $a \in D'$ (i.e., for any neighborhood V of a, one has $V \setminus \{a\} \cap D \neq \emptyset$).

The real number L is called the limit of $f(x)$ as x tends to a if

$$\forall\, \varepsilon > 0,\ \exists\, \delta = \delta(\varepsilon) > 0 \text{ s.t. } |f(x) - L| < \varepsilon,\ \forall\, x :\ 0 < \|x - a\| < \delta.$$

We write $\lim\limits_{x \to a} f(x) = L$.

Let us now consider $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ a vector valued and $a \in D'$.

The vector $L \in \mathbb{R}^m$ is called the limit of $f(x)$ as x tends to a, if

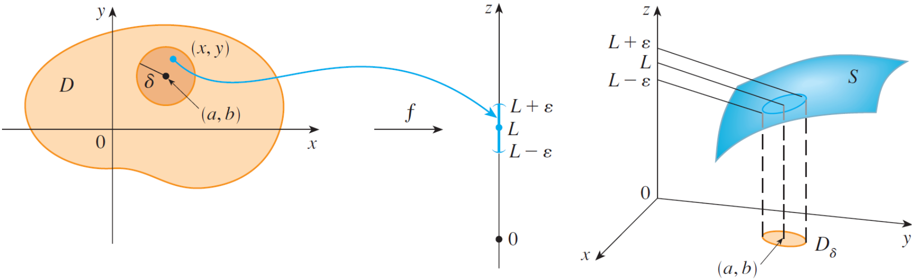
$$\forall\, \varepsilon > 0,\ \exists\, \delta = \delta(\varepsilon) > 0 \text{ s.t. } \|f(x) - L\| < \varepsilon,\ \forall\, x :\ 0 < \|x - a\| < \delta.$$

We write $\lim\limits_{x \to a} f(x) = L$.

LIMITS

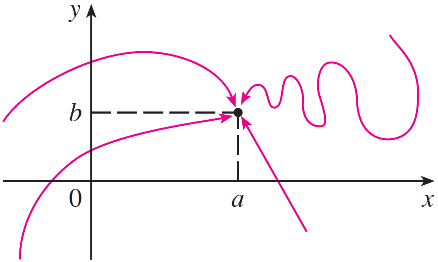
For functions of two real variables:

$\lim_{(x,y)\to(a,b)} f(x,y) = L$ if the values of $f(x,y)$ approach the number L as the point (x,y) approaches the point (a,b) along any path that stays within the domain of f.



Limits of functions of two real variables

For functions of two variables we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever as long as (x, y) stays within the domain of f.



If

- $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and
- $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, with $L_1 \neq L_2$,

then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Examples

Example 1. The limit of $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$ at $(0,0)$.

Using the substitution $u = x^2 + y^2$ and the remarkable limit, we have:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{u\to 0} \frac{\sin(u)}{u} = 1.$$

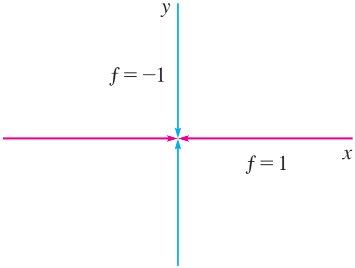
Example 2. The function $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ does not have a limit at $(0,0)$:

Along the x-axis: $Ox: \; y = 0$

$$f(x,0) = \frac{x^2}{x^2} = 1 \xrightarrow{x\to 0} 1$$

Along the y-axis: $Oy: \; x = 0$

$$f(0,y) = \frac{-y^2}{y^2} = -1 \xrightarrow{y\to 0} -1$$



Examples

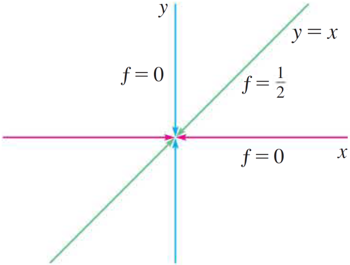
Example 3. The function $f(x,y) = \dfrac{xy}{x^2+y^2}$ does not have a limit at $(0,0)$:

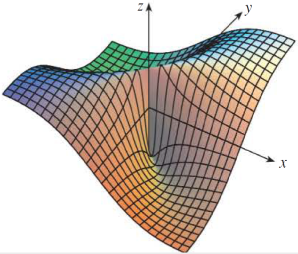
Along the x-axis: $Ox: \; y = 0$

$$f(x,0) = \frac{0}{x^2} = 0 \xrightarrow{x\to 0} 0$$

Along the first bisector: $y = x$

$$f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2} \xrightarrow{x\to 0} \frac{1}{2}$$





Important Properties

- If $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$ and $L = (L_1, \dots, L_m)$, then $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a} f_i(x) = L_i$ for any $i = \overline{1, m}$.
- The same limit laws as for functions of one real variable.
- Heine's criterion for the limit
  The function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ has a limit as x approaches a if and only if for any sequence (x_k), $x_k \in D$, $x_k \neq a$, and $x_k \to a$ as $k \to \infty$, the sequence $(f(x_k))$ converges.
- Cauchy-Bolzano's criterion for the limit
  The function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ has a limit as $x \to a$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < \|x' - a\| < \delta$ and $0 < \|x'' - a\| < \delta$ then $\|f(x') - f(x'')\| < \varepsilon$.

CONTINUITY

A function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in D$ if $\lim_{x \to a} f(x) = f(a)$.

Rules for continuity:

- If the real valued functions of n variables f and g are continuous at a then so are $f + g$, $f \cdot g$ and $\frac{1}{f}$.
- If $f : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m$ is continuous at $a \in A$ and $g : B \subset \mathbb{R}^m \to \mathbb{R}^p$ is continuous at $f(a) = b \in \mathbb{R}^m$, then the composite function $g \circ f : A \to \mathbb{R}^p$ is continuous at a.

A function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous (on D) if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for $x', x'' \in D$ we have that if $\|x' - x''\| < \delta$ then $\|f(x') - f(x'')\| < \varepsilon$.

Example

Example 4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

We have:

$$|f(x,y)| = 3|y| \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1} \leq 3|y| \xrightarrow{y \to 0} 0.$$

By the squeeze rule, we deduce that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

As $f(0,0) = 0 = \lim\limits_{(x,y)\to(0,0)} f(x,y)$, it follows that f is continuous at $(0,0)$.

As $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$, for $(x,y) \neq (0,0)$, the function f is continuous at every point $(x,y) \neq (0,0)$. $\implies$ f is continuous on $\mathbb{R}^2$.

Important properties

Continuity of the scalar components:

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = (f_1(x), \dots, f_m(x))$ and $a \in D$.
The function f is continuous at $a \in D$ if and only if the scalar components f_i, $i = 1, 2, \dots, m$ are continuous at a.

Heine's criterion for continuity

The function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in A$ if and only if for any sequence $(x_k) \subset D$ which converges to a, the sequence $(f(x_k))$ converges to $f(a)$.

Cauchy-Bolzano's criterion for continuity

The function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in D$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x' - a\| < \delta$ and $\|x'' - a\| < \delta$ then $\|f(x') - f(x'')\| < \varepsilon$.

The boundedness property

If $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous on the compact (closed and bounded) set D, then the set $f(D)$ is bounded and there exists $a \in D$ such that $\|f(a)\| = \sup \|f(D)\|$.